Algorithmic Number Theory and Public-key Cryptography

Course 3

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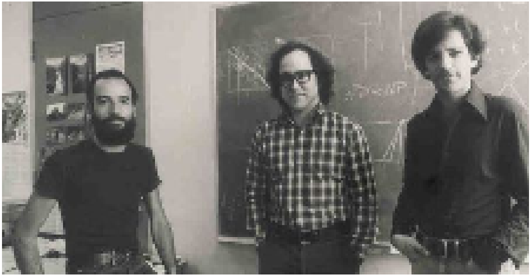
The RSA algorithm

- The RSA algorithm is the most widely-used public-key encryption algorithm
  - Invented in 1977 by Rivest, Shamir and Adleman.
  - Used for encryption and signature.
  - Widely used in electronic commerce protocols (SSL).



Public-key encryption

- Public-key encryption: two keys.
  - One key is made public and used to encrypt.
  - The other key is kept private and enables to decrypt.
- Alice wants to send a message to Bob:
  - She encrypts it using Bob's public-key.
  - Only Bob can decrypt it using his own private-key.
  - Alice and Bob do not need to meet to establish a secure communication.
- Security:
  - It must be difficult to recover the private-key from the public-key
  - but not enough in practice.

- Key generation:
  - Generate two large distinct primes p and q of same bit-size.
  - Compute $n = p \cdot q$ and $\phi = (p-1)(q-1)$.
  - Select a random integer e, $1 < e < \phi$ such that $\gcd(e, \phi) = 1$
  - Compute the unique integer d such that

$$e \cdot d \equiv 1 \pmod{\phi}$$

  using the extended Euclidean algorithm.
  - The public key is (n, e). The private key is d.

RSA encryption

- Encryption
  - Given a message $m \in [0, n-1]$ and the recipent's public-key (n, e), compute the ciphertext:

$$c = m^e \mod n$$

- Decryption
  - Given a ciphertext c, to recover m, compute:

$$m = c^d \mod n$$

- Message encoding
  - The message m is viewed as an integer between 0 and $n - 1$
  - One can always interpret a bit-string of length less than $\lfloor \log_2 n \rfloor$ as such a number.
  - One must be careful: plain RSA encryption is insecure.

Euler function

- Definition:
  - $\phi(n)$ for $n > 0$ is defined as the number of integers a comprised between 0 and $n-1$ such that $\gcd(a,n) = 1$.
  - $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$.
- Equivalently:
  - Let $\mathbb{Z}_n^*$ be the set of integers a comprised between 0 and $n-1$ such that $\gcd(a,n) = 1$.
  - Then $\phi(n) = |\mathbb{Z}_n^*|$.

Properties

- If $p \geq 2$ is prime, then

$$\phi(p) = p - 1$$

- More generally, for any $e \geq 1$,

$$\phi(p^e) = p^{e-1} \cdot (p - 1)$$

- For $n, m > 0$ such that $\gcd(n, m) = 1$, we have:

$$\phi(n \cdot m) = \phi(n) \cdot \phi(m)$$

Euler's theorem

- Theorem
  - For any integer $n > 1$ and any integer a such that $\gcd(a, n) = 1$, we have $a^{\phi(n)} \equiv 1 \mod n$.
- Proof
  - Consider the map $f : \mathbb{Z}_n^* \to \mathbb{Z}_n^*$, such that $f(b) = a \cdot b$ for any $b \in \mathbb{Z}^*$.
  - f is a permutation, therefore :

$$\prod_{b \in \mathbb{Z}_n^*} b = \prod_{b \in \mathbb{Z}_n^*} (a \cdot b) = a^{\phi(n)} \cdot \left(\prod_{b \in \mathbb{Z}_n^*} b \right)$$

  - Therefore, we obtain $a^{\phi(n)} \equiv 1 \mod n$.

Fermat's little theorem

- Theorem
  - For any prime p and any integer $a \neq 0 \mod p$, we have $a^{p-1} \equiv 1 \mod p$. Moreover, for any integer a, we have $a^p \equiv a \mod p$.
- Proof
  - Follows from Euler's theorem and $\phi(p) = p - 1$.

Proof that decryption works

- We must show that $m^{ed} = m \bmod n$.
- Since $e \cdot d \equiv 1 \mod \phi$, there is an integer k such that $e \cdot d = 1 + k \cdot \phi = 1 + k \cdot (p-1) \cdot (q-1)$. Therefore we must show that:

$$m^{1+k\cdot(p-1)\cdot(q-1)} \equiv m \pmod n$$

- If $m \neq 0 \mod p$, then by Fermat's little theorem $m^{p-1} \equiv 1 \pmod p$, which gives :

$$m^{1+k\cdot(p-1)\cdot(q-1)} \equiv m \pmod p$$

  - This equality is also true if $m \equiv 0 \pmod p$.
  - This gives $m^{ed} \equiv m \pmod p$ for all m.
  - Similarly, $m^{ed} \equiv m \pmod q$ for all m.
  - By the Chinese Remainder Theorem, if $p \neq q$, then

$$m^{ed} \equiv m \pmod n$$

Decrypting with CRT

- Given the factors p and q of $n = p \cdot q$, instead of computing $m = c^d \bmod n$, compute:
  - $m_p = c^{d_p} \bmod p$, where $d_p = d \bmod (p-1)$
  - $m_q = c^{d_q} \bmod q$, where $d_q = d \bmod (q-1)$
  - Using CRT, find m such that $m \equiv m_p \pmod{p}$ and $m \equiv m_q \pmod{q}$:

$$m = \left(m_p \cdot (q^{-1} \bmod p) \cdot q + m_q \cdot (p^{-1} \bmod q) \cdot p\right) \bmod n$$

- Since exponentiation is cubic, this is roughly 4 times faster.

Security of RSA

- The security of RSA is based on the hardness of factoring.
  - Given $n = p \cdot q$, it should be difficult to recover p and q.
  - No efficient algorithm is known to do that. Best algorithms have sub-exponential complexity.
  - Factoring record: a 768-bit RSA modulus n.
  - In practice, one uses at least 1024-bit RSA moduli.
- However, there are many other lines of attacks.
  - Attacks against plain RSA encryption
  - Low private / public exponent attacks
  - Implementation attacks: timing attacks, power attacks and fault attacks

The RSA signature scheme

- Key generation :
  - Public modulus: $N = p \cdot q$ where p and q are large primes.
  - Public exponent : e
  - Private exponent: d, such that $d \cdot e = 1 \mod \phi(N)$
- To sign a message m, the signer computes :
  - $s = m^d \mod N$
  - Only the signer can sign the message.
- To verify the signature, one checks that:
  - $m = s^e \mod N$
  - Anybody can verify the signature

Hash-and-sign paradigm

- There are many attacks on basic RSA signatures:
  - Existential forgery: $r^e = m \mod N$
  - Chosen-message attack: $(m_1 \cdot m_2)^d = m_1^d \cdot m_2^d \mod N$
- To prevent from these attacks, one usually uses a hash function. The message is first hashed, then padded.
  - $m \longrightarrow H(m) \longrightarrow \texttt{1001}\ldots\texttt{0101}\|H(m)$
  - Example: PKCS#1 v1.5: $\mu(m) = \texttt{0001 FF}\ldots\texttt{FF00}||c_{\mathrm{SHA}}||\mathrm{SHA}(m)$
  - ISO 9796-2: $\mu(m) = \texttt{6A}\|m[1]\|H(m)\|\texttt{BC}$
  - The signature is then $\sigma = \mu(m)^d \mod N$

Attacks against RSA

- Factoring
  - Equivalence between factoring and breaking RSA ?
- Mathematical attacks
  - Attacks against plain RSA encryption and signature
  - Heuristic countermeasures
  - Low private / public exponent attacks
  - Provably secure constructions
- Implementation attacks
  - Timing attacks, power attacks and fault attacks
  - Countermeasures

Factoring attack

- Factoring large integers
  - Best factoring algorithm: Number Field Sieve
  - Sub-exponential complexity

$$\exp\left((c + \circ(1))\, n^{1/3} \log^{2/3} n\right)$$

  for n-bit integer.
  - Current factoring record: 768-bit RSA modulus.
- Use at least 1024-bit RSA moduli
  - 2048-bit for long-term security.

Factoring vs breaking RSA

- Breaking RSA:
  - Given (N, e) and y, find x such that $y = x^e \mod N$
- Open problem
  - Is breaking RSA equivalent to factoring ?
- Knowing d is equivalent to factoring
  - Probabilistic algorithm (RSA, 1978)
  - Deterministic algorithm (A. May 2004, J.S. Coron and A. May 2007)

Elementary attacks

- Plain RSA encryption: dictionary attack
  - If only two possible messages m_0 and m_1, then only $c_0 = (m_0)^e \mod N$ and $c_1 = (m_1)^e \mod N$.
  - $\Rightarrow$ encryption must be probabilistic.
- PKCS#1 v1.5
  - $\mu(m) = 0002\|r\|00\|m$
  - $c = \mu(m)^e \mod N$
  - Still insufficient (Bleichenbacher's attack, 1998)

Attacks against Plain RSA signature

- Existential forgery
  - $r^e = m \mod N$, so r is signature of m
- Chosen message attack
  - $(m_1 \cdot m_2)^d = m_1^d \cdot m_2^d \mod N$
- To prevent from these attacks, one first computes $\mu(m)$, and lets $s = \mu(m)^d \mod N$
  - ISO 9796-1:

$$\mu(m) = \bar{s}(m_z)s(m_{z-1})m_z m_{z-1} \dots s(m_1)s(m_0)m_0 6$$

  - ISO 9796-2:

$$\mu(m) = \mathtt{6A} \| m[1] \| H(m) \| \mathtt{BC}$$

  - PKCS#1 v1.5:

$$\mu(m) = \mathtt{0001\ FF\dots FF00} || c_{\mathrm{SHA}} || \mathrm{SHA}(m)$$

Attacks against RSA signatures

- Desmedt and Odlyzko attack (Crypto 85)
  - Based on finding messages m such that $\mu(m)$ is smooth (product of small primes only)
  - $\mu(m_i) = \prod_j p_j^{\alpha_{i,j}}$ for many messages m_i.
  - Solve a linear system and write $\mu(m_k) = \prod_i \mu(m_i)$
  - Then $\mu(m_k)^d = \prod_i \mu(m_i)^d \mod N$
- Application to ISO 9796-1 and ISO 9796-2 signatures
  - Cryptanalysis of ISO 9796-1 (Coron, Naccache, Stern, 1999)
  - Cryptanalysis of ISO 9796-2 (Coron, Naccache, Tibouchi, Weinmann, 2009)
  - Extension of Desmedt and Odlyzko attack.
  - For ISO 9796-2 the attack is feasible if the output size of the hash function is small enough.

Low private exponent attacks

- To reduce decryption time, one could use a small d
  - Wiener's attack: recover d if $d < N^{0.25}$
- Boneh and Durfee's attack (1999)
  - Recover d if $d < N^{0.29}$
  - Based on lattice reduction and Coppersmith's technique
  - Open problem: extend to $d < N^{0.5}$
- Conclusion: devastating attack
  - Use a full-size d

Low public exponent attack

- To reduce encryption time, one can use a small e
  - For example $e = 3$ or $e = 2^{16} + 1$
- Coppersmith's theorem :
  - Let N be an integer and f be a polynomial of degree δ. Given N and f, one can recover in polynomial time all x_0 such that $f(x_0) = 0 \mod N$ and $x_0 < N^{1/\delta}$.
- Application: partially known message attack :
  - If $c = (B\|m)^3 \mod N$, one can recover m if $|m| < |N|/3$
  - Define $f(x) = (B \cdot 2^k + x)^3 - c \mod N$.
  - Then $f(m) = 0 \mod N$ and apply Coppersmith's theorem to recover m.

Low public exponent attack

- Coppersmith's short pad attack
  - Let $c_1 = (m \| r_1)^3 \mod N$ and $c_2 = (m \| r_2)^3 \mod N$
  - One can recover m if $r_1, r_2 < N^{1/9}$
  - Let $g_1(x, y) = x^3 - c_1$ and $g_2(x, y) = (x + y)^3 - c_2$.
  - g_1 and g_2 have a common root $(m \| r_1, r_2 - r_1)$ modulo N.
  - $h(y) = \mathrm{Res}_x(g_1, g_2)$ has a root $\Delta = r_2 - r_1$, with $\deg h = 9$.
  - To recover $m \| r_1$, take gcd of $g_1(x, \Delta)$ and $g_2(x, \Delta)$.
- Conclusion:
  - Attack only works for particular encryption schemes.
  - Low public exponent is secure when provably secure construction is used. One often takes $e = 2^{16} + 1$.

Implementation attacks

- The implementation of a cryptographic algorithm can reveal more information
- Passive attacks :
  - Timing attacks (Kocher, 1996): measure the execution time
  - Power attacks (Kocher et al., 1999): measure the power consumption
- Active attacks :
  - Fault attacks (Boneh et al., 1997): induce a fault during computation
  - Invasive attacks: probing.

Timing attacks

- Described on RSA by Kocher at Crypto 96.
  - Let $d = \sum_{i=0}^{n} 2^i d_i$.
  - Computing $m^d \mod N$ using square and multiply :
    - Let $z \leftarrow m$
      For $i = n-1$ downto 0 do
      Let $z \leftarrow z^2 \mod N$
      If $d_i = 1$ let $z \leftarrow z \cdot m \mod N$
- Attack
  - Let T_i be the total time needed to compute $m_i^d \mod N$
  - Let t_i be the time needed to compute $m_i^3 \mod N$
  - If $d_{n-1} = 1$, the variables t_i and T_i are correlated, otherwise they are independent. This gives d_{n-1}.

Countermeasures

- Implement in constant time
  - Not always possible with hardware crypto-processors.
- Exponent blinding:
  - Compute $m^{d+k\cdot\phi(N)} = m^d \mod N$ for random k.
- Message blinding
  - Compute $(m \cdot r)^d / r^d = m^d \mod N$ for random r.
- Modulus randomization
  - Compute $m^d \mod (N \cdot r)$ and reduce modulo N.
- or a combination of the three.

Power attacks

- Based on measuring power consumption
  - Introduced by Kocher *et al.* at Crypto 99.
  - Initially applied on DES, but any cryptographic algorithm is vulnerable.
- Attack against exponentiation $m^d \mod N$:
  - If power consumption correlated with some bits of $m^3 \mod N$, this means that $m^3 \mod N$ was effectively computed, and so $d_{n-1} = 1$.
  - Enables to recover d_{n-1} and by recursion the full d.

Countermeasures

- Hardware countermeasures
  - Constant power consumption; dual rail logic.
  - Random delays to desynchronise signals.
- Software countermeasures
  - Same as for timing attacks
  - Goal: randomization of execution
  - Drawback: increases execution time.

Fault attacks

- Induce a fault during computation
  - By modifying voltage input
- RSA with CRT: to compute $s = m^d \mod N$, compute :
  - $s_p = m^{d_p} \mod p$ where $d_p = d \mod p - 1$
  - $s_q = m^{d_q} \mod q$ where $d_q = d \mod q - 1$
  - and recombine s_p and s_q using CRT to get $s = m^d \mod N$
- Fault attack against RSA with CRT (Boneh *et al.*, 1996)
  - If s_p is incorrect, then $s^e \neq m \mod N$ while $s^e = m \mod q$
  - Therefore, $\gcd(N, s^e - m)$ gives the prime factor q.

Possible projects

- Implementation of RSA: big integer library.
- Factoring algorithms. Implementation of Pollard's rho algorithm or quadratic sieve
- Implementation attacks against RSA. Simulation of a side-channel attack.